

# FEKETE-SZEGÖ PROBLEM FOR CERTAIN CLASSES OF MA-MINDA BI-UNIVALENT FUNCTIONS

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**ABSTRACT.** In the present work, we propose to investigate the Fekete-Szegő inequalities certain classes of analytic and bi-univalent functions defined by subordination. The results in the bounds of the third coefficient which improve many known results concerning different classes of bi-univalent functions. Some interesting applications of the results presented here are also discussed.

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## 1. INTRODUCTION

Let  $\mathcal{A}$  denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disc  $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ . Further, by  $\mathcal{S}$  we will show the family of all functions in  $\mathcal{A}$  which are univalent in  $\mathbb{U}$ .

For two functions  $f$  and  $g$ , analytic in  $\mathbb{U}$ , we say that the function  $f(z)$  is subordinate to  $g(z)$  in  $\mathbb{U}$ , and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U})$$

if there exists a Schwarz function  $w(z)$ , analytic in  $\mathbb{U}$ , with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

In particular, if the function  $g$  is univalent in  $\mathbb{U}$ , the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

It is well known that every function  $f \in \mathcal{S}$  has an inverse  $f^{-1}$ , defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left( |w| < r_0(f); \quad r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (1.2)$$

A function  $f \in \mathcal{A}$  is said to be bi-univalent in  $\mathbb{U}$  if both  $f(z)$  and  $f^{-1}(z)$  are univalent in  $\mathbb{U}$ . Let  $\Sigma$  denote the class of bi-univalent functions in  $\mathbb{U}$  given by (1.1). For a brief history and interesting examples of functions which are in (or which are not in) the class  $\Sigma$ , together with various other properties of the bi-univalent function class  $\Sigma$  one can

refer the work of Srivastava et al. [15] and references therein. In fact, the study of the coefficient problems involving bi-univalent functions was reviewed recently by Srivastava et al. [15]. Various subclasses of the bi-univalent function class  $\Sigma$  were introduced and non-sharp estimates on the first two coefficients  $|a_2|$  and  $|a_3|$  in the Taylor-Maclaurin series expansion (1.1) were found in several recent investigations (see, for example, [1, 2, 3, 4, 5, 6, 8, 9, 12, 13, 14, 16, 18, 19]). The aforementioned all these papers on the subject were actually motivated by the pioneering work of Srivastava et al. [15]. However, the problem to find the coefficient bounds on  $|a_n|$  ( $n = 3, 4, \dots$ ) for functions  $f \in \Sigma$  is still an open problem.

Some of the important and well-investigated subclasses of the univalent function class  $\mathcal{S}$  include (for example) the class  $\mathcal{S}^*(\alpha)$  of starlike functions of order  $\alpha$  in  $\mathbb{U}$  and the class  $\mathcal{K}(\alpha)$  of convex functions of order  $\alpha$  in  $\mathbb{U}$ . By definition, we have

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha; z \in \mathbb{U}; 0 \leq \alpha < 1 \right\} \quad (1.3)$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha; z \in \mathbb{U}; 0 \leq \alpha < 1 \right\}. \quad (1.4)$$

For  $0 \leq \alpha < 1$ , a function  $f \in \Sigma$  is in the class  $\mathcal{S}_\Sigma^*(\alpha)$  of bi-starlike function of order  $\alpha$ , or  $\mathcal{K}_{\Sigma,\alpha}$  of bi-convex function of order  $\alpha$  if both  $f$  and  $f^{-1}$  are respectively starlike or convex functions of order  $\alpha$ . For  $0 < \beta \leq 1$ , a function  $f \in \Sigma$  is strongly bi-starlike function of order  $\beta$ , if both the functions  $f$  and  $f^{-1}$  are strongly starlike of order  $\beta$ . We denote the class of all such functions is denoted by  $\mathcal{S}_{\Sigma,\beta}^*$ .

Let  $\varphi$  be an analytic and univalent function with positive real part in  $\mathbb{U}$  with  $\varphi(0) = 1$ ,  $\varphi'(0) > 0$  and  $\varphi$  maps the unit disk  $\mathbb{U}$  onto a region starlike with respect to 1, and symmetric with respect to the real axis. The Taylor's series expansion of such function is of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots, \quad (1.5)$$

where all coefficients are real and  $B_1 > 0$ . Throughout this paper we assume that the function  $\varphi$  satisfies the above conditions one or otherwise stated.

By  $\mathcal{S}^*(\varphi)$  and  $\mathcal{K}(\varphi)$  we denote the following classes of functions

$$\mathcal{S}^*(\varphi) := \left\{ f : f \in \mathcal{A} \text{ and } \frac{zf'(z)}{f(z)} \prec \varphi(z); z \in \mathbb{U} \right\} \quad (1.6)$$

and

$$\mathcal{K}(\varphi) := \left\{ f : f \in \mathcal{A} \text{ and } 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z); z \in \mathbb{U} \right\}. \quad (1.7)$$

The classes  $\mathcal{S}^*(\varphi)$  and  $\mathcal{K}(\varphi)$  are the extensions of a classical sets of a starlike and convex functions and in a such form were defined and studied by Ma and Minda [7]. A function  $f$  is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both  $f$  and  $f^{-1}$  are respectively Ma-Minda starlike or convex. These classes are denoted respectively by  $\mathcal{S}_\Sigma^*(\varphi)$  and  $\mathcal{K}_\Sigma(\varphi)$  (see [1]).

In order to derive our main results, we will need the following lemma.

**Lemma 1.1.** (see [11]) *If  $p \in \mathcal{P}$ , then  $|p_i| \leq 2$  for each  $i$ , where  $\mathcal{P}$  is the family of all functions  $p$ , analytic in  $\mathbb{U}$ , for which*

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}),$$

where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}).$$

Motivated by the aforementioned works (especially [20] and [3, 10, 14]), we consider the following subclass of the function class  $\Sigma$  (see also, [17]).

A function  $f \in \Sigma$  given by (1.1) is said to be in the class  $\mathcal{N}_\Sigma^{\mu,\lambda}(\varphi)$  if the following conditions are satisfied:

$$(1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} \prec \varphi(z) \quad (\lambda \geq 1, \mu \geq 0, z \in \mathbb{U}) \quad (1.8)$$

and

$$(1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} \prec \varphi(w) \quad (\lambda \geq 1, \mu \geq 0, w \in \mathbb{U}), \quad (1.9)$$

where  $g(w) = f^{-1}(w)$ .

*Remark 1.2.* From among the many choices of  $\mu$ ,  $\lambda$  and the function  $\varphi$  which would provide the following known subclasses:

- (1)  $\mathcal{N}_\Sigma^{1,1}(\varphi) = \mathcal{H}_\Sigma^\varphi$  [1, p.345].
  - (2)  $\mathcal{N}_\Sigma^{1,1}\left(\left(\frac{1+z}{1-z}\right)^\beta\right) = \mathcal{H}_\Sigma^\beta$  ( $0 < \beta \leq 1$ ) and  $\mathcal{N}_\Sigma^{1,1}\left(\frac{1+(1-2\alpha)z}{1-z}\right) = \mathcal{H}_\Sigma^\alpha$  ( $0 \leq \alpha < 1$ ) [15, Definitions 1 and 2].
  - (3)  $\mathcal{N}_\Sigma^{1,\lambda}(\varphi) = \mathcal{R}_\Sigma(\lambda, \varphi)$  ( $\lambda \geq 0$ ) [12, Definition 1.1].
  - (4)  $\mathcal{N}_\Sigma^{1,\lambda}\left(\left(\frac{1+z}{1-z}\right)^\beta\right) = \mathcal{B}_\Sigma(\beta, \lambda)$  ( $\lambda \geq 1; 0 < \beta \leq 1$ ) and  $\mathcal{N}_\Sigma^{1,\lambda}\left(\frac{1+(1-2\alpha)z}{1-z}\right) = \mathcal{B}_\Sigma(\alpha, \lambda)$  ( $\lambda \geq 1; 0 \leq \alpha < 1$ ) [5, Definitions 2.1 and 3.1].
  - (5)  $\mathcal{N}_\Sigma^{\mu,1}(\varphi) = \mathcal{F}_\Sigma^\mu(\varphi)$  ( $\mu \geq 0$ ) [12, Definition 2.1].
  - (6)  $\mathcal{N}_\Sigma^{0,1}\left(\left(\frac{1+z}{1-z}\right)^\beta\right) = \mathcal{S}_{\Sigma,\beta}^*$  ( $0 < \beta \leq 1$ ) and  $\mathcal{N}_\Sigma^{0,1}\left(\frac{1+(1-2\alpha)z}{1-z}\right) = \mathcal{S}_\Sigma^*(\alpha)$  ( $0 \leq \alpha < 1$ ).
  - (7)  $\mathcal{N}_\Sigma^{\mu,\lambda}\left(\left(\frac{1+z}{1-z}\right)^\beta\right) = \mathcal{N}_\Sigma^{\mu,\lambda}(\beta)$  ( $\lambda \geq 1; \mu \geq 0; 0 < \beta \leq 1$ ) [3, Definitions 2.1].
- and
- $$\mathcal{N}_\Sigma^{\mu,\lambda}\left(\frac{1+(1-2\alpha)z}{1-z}\right) = \mathcal{N}_\Sigma^{\mu,\lambda}(\alpha) \quad (\lambda \geq 1; \mu \geq 0; 0 \leq \alpha < 1) \quad [3, \text{Definitions 3.1}].$$

In this paper we shall obtain the Fekete-Szegő inequalities for  $\mathcal{N}_\Sigma^{\mu,\lambda}(\varphi)$  and its special classes. These inequalities will result in bounds of the third coefficient which are, in some cases, better than these obtained in [1, 3, 5, 14, 15, 17].

## 2. MAIN RESULTS

**Theorem 2.1.** *Let  $f$  of the form (1.1) be in  $\mathcal{N}_\Sigma^{\mu,\lambda}(\varphi)$  and  $\delta \in \mathbb{R}$ . Then*

$$|a_3 - \delta a_2^2| \leq \begin{cases} \frac{B_1}{2\lambda + \mu} & ; |\delta - 1| \leq \frac{\mu+1}{2} \left| 1 + \frac{2(B_1-B_2)(\lambda+\mu)^2}{B_1^2(2\lambda+\mu)(1+\mu)} \right| \\ \frac{2B_1^3|\delta-1|}{|(2\lambda+\mu)(1+\mu)B_1^2+2(B_1-B_2)(\lambda+\mu)^2|} & ; |\delta - 1| \geq \frac{\mu+1}{2} \left| 1 + \frac{2(B_1-B_2)(\lambda+\mu)^2}{B_1^2(2\lambda+\mu)(1+\mu)} \right|. \end{cases} \quad (2.1)$$

*Proof.* Since  $f \in \mathcal{N}_\Sigma^{\mu,\lambda}(\varphi)$ , there exists two analytic functions  $r, s : \mathbb{U} \rightarrow \mathbb{U}$ , with  $r(0) = 0 = s(0)$ , such that

$$(1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} = \varphi(r(z)) \quad (2.2)$$

and

$$(1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} = \varphi(s(z)). \quad (2.3)$$

Define the functions  $p$  and  $q$  by

$$p(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots \quad (2.4)$$

and

$$q(z) = \frac{1 + s(z)}{1 - s(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \dots \quad (2.5)$$

or equivalently,

$$r(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left( p_1 z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \left( p_3 + \frac{p_1}{2} \left( \frac{p_1^2}{2} - p_2 \right) - \frac{p_1 p_2}{2} \right) z^3 + \dots \right) \quad (2.6)$$

and

$$s(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left( q_1 z + \left( q_2 - \frac{q_1^2}{2} \right) z^2 + \left( q_3 + \frac{q_1}{2} \left( \frac{q_1^2}{2} - q_2 \right) - \frac{q_1 q_2}{2} \right) z^3 + \dots \right). \quad (2.7)$$

Using (2.6) and (2.7) in (2.2) and (2.3), we have

$$(1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} = \varphi \left( \frac{p(z) - 1}{p(z) + 1} \right) \quad (2.8)$$

and

$$(1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} = \varphi \left( \frac{q(w) - 1}{q(w) + 1} \right). \quad (2.9)$$

Again using (2.6) and (2.7) along with (1.5), it is evident that

$$\varphi \left( \frac{p(z) - 1}{p(z) + 1} \right) = 1 + \frac{1}{2} B_1 p_1 z + \left( \frac{1}{2} B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} B_2 p_1^2 \right) z^2 + \dots \quad (2.10)$$

and

$$\varphi \left( \frac{q(w) - 1}{q(w) + 1} \right) = 1 + \frac{1}{2} B_1 q_1 w + \left( \frac{1}{2} B_1 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} B_2 q_1^2 \right) w^2 + \dots \quad (2.11)$$

It follows from (2.8), (2.9), (2.10) and (2.11) that

$$(\lambda + \mu) a_2 = \frac{1}{2} B_1 p_1 \quad (2.12)$$

$$(2\lambda + \mu) \left[ a_3 + \frac{a_2^2}{2} (\mu - 1) \right] = \frac{1}{2} B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} B_2 p_1^2 \quad (2.13)$$

$$- (\lambda + \mu) a_2 = \frac{1}{2} B_1 q_1 \quad (2.14)$$

and

$$(2\lambda + \mu) \left[ \frac{a_2^2}{2} (\mu + 3) - a_3 \right] = \frac{1}{2} B_1 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} B_2 q_1^2. \quad (2.15)$$

From (2.12) and (2.14), we find that

$$a_2 = \frac{B_1 p_1}{2(\lambda + \mu)} = \frac{-B_1 q_1}{2(\lambda + \mu)} \quad (2.16)$$

it follows that

$$p_1 = -q_1 \quad (2.17)$$

and

$$8(\lambda + \mu)^2 a_2^2 = B_1^2(p_1^2 + q_1^2). \quad (2.18)$$

Adding (2.13) and (2.15), we have

$$a_2^2(2\lambda + \mu)(\mu + 1) = \frac{B_1}{2}(p_2 + q_2) + \frac{(B_2 - B_1)}{4}(p_1^2 + q_1^2). \quad (2.19)$$

Substituting (2.16) and (2.17) into (2.19), we get,

$$p_1^2 = \frac{B_1 2(\lambda + \mu)^2(p_2 + q_2)}{B_1^2(2\lambda + \mu)(\mu + 1) - 2(B_2 - B_1)(\lambda + \mu)^2}. \quad (2.20)$$

Now, (2.16) and (2.20) yield

$$a_2^2 = \frac{B_1^3(p_2 + q_2)}{2(\mu + 1)(2\lambda + \mu)B_1^2 + 4(B_1 - B_2)(\lambda + \mu)^2}. \quad (2.21)$$

By subtracting (2.13) from (2.15) and a computation using (2.17) finally lead to

$$a_3 = a_2^2 + \frac{B_1(p_2 - q_2)}{8\lambda + 4\mu}. \quad (2.22)$$

From (2.21) and (2.22) it follows that

$$a_3 - \delta a_2^2 = B_1 \left[ \left( h(\delta) + \frac{1}{8\lambda + 4\mu} \right) p_2 + \left( h(\delta) - \frac{1}{8\lambda + 4\mu} \right) q_2 \right],$$

where

$$h(\delta) = \frac{B_1^2(1 - \delta)}{2(\mu + 1)(2\lambda + \mu)B_1^2 + 4(B_1 - B_2)(\lambda + \mu)^2}.$$

Since all  $B_j$  are real and  $B_1 > 0$ , we conclude that

$$|a_3 - \delta a_2^2| \leq \begin{cases} \frac{B_1}{2\lambda + \mu} & ; 0 \leq |h(\delta)| < \frac{1}{8\lambda + \mu} \\ 4B_1|h(\delta)| & ; |h(\delta)| \geq \frac{1}{8\lambda + \mu}, \end{cases}$$

which completes the proof.  $\square$

*Remark 2.2.* For  $\lambda = \mu = 1$  Theorem 2.1 reduces to the results discussed in [20, Theorem 1, p.172].

### 3. COROLLARIES AND CONSEQUENCES

Taking  $\delta = 1$ ,  $\delta = 0$  in Theorem 2.1, we have the following corollaries.

**Corollary 3.1.** *If  $f \in \mathcal{N}_{\Sigma}^{\mu, \lambda}(\varphi)$  then*

$$|a_3 - a_2^2| \leq \frac{B_1}{2\lambda + \mu}.$$

**Corollary 3.2.** *If  $f \in \mathcal{N}_{\Sigma}^{\mu, \lambda}(\varphi)$  then*

$$|a_3| \leq \begin{cases} \frac{B_1}{2\lambda + \mu} & ; \frac{(B_1 - B_2)}{B_1^2} \in \left( -\infty, \frac{-(3+\mu)(2\lambda + \mu)}{2(\lambda + \mu)^2} \right] \cup \left[ \frac{(1-\mu)(2\lambda + \mu)}{2(\lambda + \mu)^2}, \infty \right) \\ \frac{2B_1^3}{|(2\lambda + \mu)(1+\mu)B_1^2 + 2(B_1 - B_2)(\lambda + \mu)^2|} & ; \frac{(B_1 - B_2)}{B_1^2} \in \left[ \frac{-(3+\mu)(2\lambda + \mu)}{2(\lambda + \mu)^2}, \frac{-(1+\mu)(2\lambda + \mu)}{2(\lambda + \mu)^2} \right) \cup \left( \frac{-(1+\mu)(2\lambda + \mu)}{2(\lambda + \mu)^2}, \frac{(1-\mu)(2\lambda + \mu)}{2(\lambda + \mu)^2} \right] \end{cases}.$$

*Remark 3.3.* Corollary 3.2 provides an improvement of the estimate  $|a_3|$  obtained by Tang et al. [17, Theorem 2.1, p.3].

In view of Remark 1.2, Corollaries 3.1 and 3.2 yield the following corollaries.

**Corollary 3.4.** *If  $f \in \mathcal{N}_\Sigma^{\mu, \lambda}(\beta)$  then*

$$|a_3| \leq \frac{2\beta}{2\lambda + \mu} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{2\beta}{2\lambda + \mu}.$$

**Corollary 3.5.** *If  $f \in \mathcal{N}_\Sigma^{\mu, \lambda}(\alpha)$  then*

$$|a_3| \leq \frac{2(1 - \alpha)}{2\lambda + \mu} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{2(1 - \alpha)}{2\lambda + \mu}.$$

*Remark 3.6.* The bounds  $|a_3|$  obtained in Corollaries 3.4 and 3.5 are improvement of the bounds  $|a_3|$  estimated by Çağlar et al. [3, Theorems 2.1 and 3.1].

*Remark 3.7.* In view of Remark 1.2 the aforecited work for the subclasses  $\mathcal{H}_\Sigma^\varphi$ ,  $\mathcal{H}_\Sigma^\beta$  and  $\mathcal{H}_\Sigma^\alpha$  are coincide with the results of Zaprawa [20, Corollaries 1 to 4, p.173].

**Corollary 3.8.** *If  $f \in \mathcal{S}_\Sigma^*(\varphi)$  then*

$$|a_3 - a_2^2| \leq \frac{B_1}{2}.$$

**Corollary 3.9.** *If  $f \in \mathcal{S}_\Sigma^*(\varphi)$  then*

$$|a_3| \leq \begin{cases} \frac{B_1}{2} & ; \frac{(B_1 - B_2)}{B_1^2} \in (-\infty, -3] \cup [0, \infty) \\ \frac{B_1^3}{|B_1^2 + (B_1 - B_2)|} & ; \frac{(B_1 - B_2)}{B_1^2} \in [-2, -1) \cup (-1, 1] \end{cases}.$$

**Corollary 3.10.** *If  $f \in \mathcal{S}_{\Sigma, \beta}^*$  then*

$$|a_3| \leq \beta \quad \text{and} \quad |a_3 - a_2^2| \leq \beta.$$

**Corollary 3.11.** *If  $f \in \mathcal{S}_\Sigma^*(\alpha)$  then*

$$|a_3| \leq 1 - \alpha \quad \text{and} \quad |a_3 - a_2^2| \leq 1 - \alpha.$$

*Remark 3.12.* The inequalities estimated in Corollaries 3.9 to 3.11 are improvement of the inequalities obtained by Zaprawa [20, Corollaries 11 and 12, p.174].

**Corollary 3.13.** *If  $f \in \mathcal{R}_\Sigma(\lambda; \varphi)$  then*

$$|a_3 - a_2^2| \leq \frac{B_1}{2\lambda + 1}.$$

**Corollary 3.14.** *If  $f \in \mathcal{R}_\Sigma(\lambda; \varphi)$  then*

$$|a_3| \leq \begin{cases} \frac{B_1}{2\lambda + 1} & ; \frac{(B_1 - B_2)}{B_1^2} \in \left(-\infty, \frac{2(2\lambda + 1)}{(\lambda + 1)^2}\right] \cup [0, \infty) \\ \frac{B_1^3}{|(2\lambda + 1)B_1^2 + (B_1 - B_2)(\lambda + 1)^2|} & ; \frac{(B_1 - B_2)}{B_1^2} \in \left[\frac{2(2\lambda + 1)}{(\lambda + 1)^2}, \frac{-(2\lambda + 1)}{(\lambda + 1)^2}\right) \cup \left(\frac{-(2\lambda + 1)}{(\lambda + 1)^2}, 0\right] \end{cases}.$$

*Remark 3.15.* Corollary 3.14 provides an improvement of  $|a_3|$  obtained by Sivaprasad Kumar et al. [12, Theorem 2.1, p.3].

**Corollary 3.16.** *If  $f \in \mathcal{B}_\Sigma(\beta, \lambda)$  then*

$$|a_3| \leq \frac{2\beta}{2\lambda + 1} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{2\beta}{2\lambda + 1}.$$

**Corollary 3.17.** *If  $f \in \mathcal{B}_\Sigma(\alpha, \lambda)$  then*

$$|a_3| \leq \frac{2(1-\alpha)}{2\lambda+1} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{2(1-\alpha)}{2\lambda+1}.$$

*Remark 3.18.* The bounds  $|a_3|$  obtained in Corollaries 3.16 and 3.17 are improvement of the bounds  $|a_3|$  estimated by Frasin and Aouf [5, Theorems 2.2 and 3.2, p.1570 and 1572], respectively.

*Remark 3.19.* If we take

$$\varphi = \varphi_0 = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \dots \quad (3.1)$$

in the class  $\mathcal{N}_\Sigma^{\mu,\lambda}(\varphi)$ , we are led to the class which we denote, for convenience, by  $\mathcal{N}_\Sigma^{\mu,\lambda}(\varphi_0)$ . In particular,  $\mathcal{N}_\Sigma^{1,1}(\varphi_0) =: \mathcal{H}_\Sigma^{\varphi_0}$ ,  $\mathcal{N}_\Sigma^{0,\mu}(\varphi_0) =: \mathcal{S}_\Sigma^*(\varphi_0)$  and  $\mathcal{N}_\Sigma^{1,\lambda}(\varphi) =: \mathcal{B}_\Sigma^*(\lambda, \varphi_0)$ .

In view of Remark 3.19, the Corollaries 3.1 and 3.2 yield the following corollaries.

**Corollary 3.20.** *If  $f \in \mathcal{N}_\Sigma^{\mu,\lambda}(\varphi_0)$  then*

$$|a_3| \leq \frac{2}{2\lambda+\mu} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{2}{2\lambda+\mu}.$$

*Remark 3.21.* For  $\mu = \lambda = 1$  the estimates in Corollary 3.20 would reduce to a known result in [20, Corollary 5, p.173]

**Corollary 3.22.** *If  $f \in \mathcal{S}_\Sigma^*(\varphi_0)$  then*

$$|a_3| \leq 1 \quad \text{and} \quad |a_3 - a_2^2| \leq 1.$$

**Corollary 3.23.** *If  $f \in \mathcal{B}_\Sigma(\lambda, \varphi_0)$  then*

$$|a_3| \leq \frac{2}{2\lambda+1} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{2}{2\lambda+1}.$$

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